

# Evolutionary Game Theory on Measure Spaces: Well-Posedness

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## Abstract

An attempt is made to find a comprehensive mathematical framework in which to investigate the problems of well-posedness and asymptotic analysis for fully nonlinear evolutionary game theoretic models. The model should be rich enough to include all classical nonlinearities, e.g., Beverton-Holt or Ricker type. For several such models formulated on the space of integrable functions, it is known that as the variance of the payoff kernel becomes small the solution converges in the long term to a Dirac measure centered at the fittest strategy; thus the limit of the solution is not in the state space of integrable functions. Starting with the replicator-mutator equation and a generalized logistic equation as bases, a general model is formulated as a dynamical system on the state space of finite signed measures. Well-posedness is established, and then it is shown that by choosing appropriate payoff kernels this model includes all classical density models, both selection and mutation, and discrete and continuous strategy (trait) spaces.

**Key Words:** Evolutionary game models, selection-mutation, space of finite signed measure, well-posedness, continuous dependence.

**AMS Subject Classification:** 91A22, 34G20, 37C25, 92D25.

## 1 Introduction

Evolutionary game theory (EGT) is the creation and study of mathematical models that describe how the strategy profile in games change over time due to mutation and selection (replication). In this paper we address the problem of finding a comprehensive mathematical framework suitable for studying the problems of well-posedness and long-term solution behavior for fully nonlinear evolutionary game theoretic models. We form a unified theory for evolutionary game theory as a dynamical system on the state space of finite signed Borel

measures under the weak star topology. In this theory, we unify the discrete and continuous strategy (trait) spaces and the pure replicator and replicator-mutator dynamics under one model.

A natural question to ask is why the formulation of a dynamical system on the state space of finite signed Borel measures under the weak star topology? Why isn't the existing mathematical machinery adequate? The next two examples will illustrate the need for such a formulation. First, we consider the following EGT model of generalized logistic growth with pure selection (i.e., strategies replicate themselves exactly and no mutation occurs) which was developed and analyzed in [4]:

$$\frac{d}{dt}x(t, q) = x(t, q)(q_1 - q_2 X(t)), \quad (1)$$

where  $X(t) = \int_Q x(t, q) dq$  is the total population,  $Q \subset \text{int}(\mathbb{R}_+^2)$  is compact and the state space is the set of continuous real valued functions  $C(Q)$ . Each  $q = (q_1, q_2) \in Q$  is a two tuple where  $q_1$  is an intrinsic replication rate and  $q_2$  is an intrinsic mortality rate. The solution to this model converges to a Dirac mass centered at the fittest  $q$ -class. This is the class with the highest birth to death ratio  $\frac{q_1}{q_2}$ , and this convergence is in a topology called *weak\** (point wise convergence of functions) [4]. However, this Dirac limit is not in the state space as it is not a continuous function. It is a measure. Thus, under this formulation one cannot treat this Dirac mass as an equilibrium (a constant) solution and hence the study of linear stability analysis is not possible. Other examples for models developed on classical state spaces such as  $L^1(X, \mu)$  that demonstrate the emergence of Dirac measures in the asymptotic limit from smooth initial densities are given in [2, 4, 10, 11, 15, 27, 26, 28]. In particular, how the measures arise naturally in a biological and adaptive dynamics environment is illustrated quite well in [26, chpt.2]. These examples show that the chosen state space for formulating such selection-mutation models must **contain** densities and Dirac masses and the topology used must **contain the ability to demonstrate convergence** of densities to Dirac masses.

The first example above assumes a continuous strategy space  $Q$  and hence the model solution is sought among density functions denoted by  $x(t, q)$ . Our second example, is the classic discrete EGT model known as the replicator-mutator equation (in this model the strategy space is assumed to be discrete). In [24, pg. 273] it is given as:

$$\dot{x}_i = \sum_{j=1}^n x_j f_j(\vec{x}) Q_{ij} - \phi(\vec{x}) x_i \quad (2)$$

where  $\vec{x} = (x_1, x_2, \dots, x_n)$  is a vector consisting of  $n$  classes each of size  $x_i$ , and  $Q_{ij}$  is the payoff kernel, i.e.,  $Q_{ij}$  is the proportion of the  $j$ -class that mutates into the  $i$ -class. Lastly  $\phi = \sum_{j=1}^n f_j x_j$  is a weighted (average) fitness. The author states that the language equation (replicator-mutator equation) is a unifying description of deterministic evolutionary dynamics. He further states that the replicator-mutator equation is used to describe the dynamics of complex adaptive systems in population dynamics, biochemistry and models of language acquisition.

Under the new formulation on the space of measures we present here, the above examples are special cases of a more general measure-valued model. In particular, with the discrete model if we allow the fitness functions  $f_j$  to be density dependent then this model can be obtained by choosing the proper initial condition composed of a linear combination of Dirac masses and the proper replication-mutation kernel which is also composed of a linear combination of Dirac masses. The example of the pure selection density model given in (1) can be realized from the measure-valued model by choosing an absolutely continuous initial measure and a continuous family of Dirac measures for the selection-mutation kernel (which represents the pure replication case). Thus, these density and discrete models can be unified under this formulation. Furthermore, our new theory combines both the pure replicator and replicator-mutator dynamics in a continuous manner. By this we mean that as the mutations get smaller and smaller the replicator-mutator model will approach the pure replicator model. This is possible because our mutation kernels are allowed to be (family of) measures as well. This presents a serious difficulty in the analysis which requires the development of some technical tools in studying the well-posedness of the new model.

Many researchers have recently devoted their attention to the study of such EGT models (e.g. [2, 4, 10, 11, 16, 17, 23, 30]). To date almost all EGT models are formulated as *density* models [4, 10, 11, 23, 30] with *linear* mutation term. There are several formulations of pure selection or replicator equation dynamics on measure spaces [2, 6, 13]. The recent formulations of selection-mutation balance equations on the probability measures by [14, 19] are novel constructions. These models describe the aging of an infinite population as a process of accumulation of mutations in a genotype. The dynamical equation which describes the system is of Kimura-Maruyama type. Thus far in selection-mutation studies the mutation process has been modeled using two different approaches: (1) A diffusion type operator [15, 30]; (2) An integral type operator that makes use of a mutation kernel [2, 10, 11, 14, 19]. Here we focus on the second approach for modeling mutation.

Perhaps the work most related to the one presented here is that in [2]. In that paper, the authors considered a *pure* selection model with density dependent birth and mortality function and a 2-dimensional trait space on the space of finite signed measures. They discussed existence-uniqueness of solutions and studied the long term behavior of the model. Here, we generalize the results in that paper in several directions. Most salient is the fact that the present paper is one in evolutionary game theory, hence the applications are possibly other than population biology. In particular, in the present paper we construct a (measure valued) EGT model. This is an ordered triple  $(Q, \mu, F)$  subject to:

$$\frac{d}{dt}\mu(t)(E) = F(\mu(t)(Q))(E), \text{ for every } E \in \mathcal{B}(Q). \quad (3)$$

Here  $Q$  is the strategy (metric) space,  $\mathcal{B}(Q)$  are the Borel sets on  $Q$ ,  $\mu(t)$  is a time dependent family of finite signed Borel measures on  $Q$  and  $F$  is a density dependent vector field such that  $\mu$  and  $F$  satisfy equation (3). The main contributions of the present work are as follows: (1) we establish well-posedness of the new measure-valued dynamical system; (2) we are able to combine models that consider both discrete and continuous parameter spaces under this formulation; no separate machinery is needed for each; (3) we are able to include both selection and mutation in one model because our setup allows for choosing the mutation to be a family of measures; (4) unlike the linear mutation term commonly used in the literature, we allow for nonlinear (density dependent) mutation term that contain all classical nonlinearities, e.g., Ricker, Beverton-Holt, Logistic; (5) unlike the one or two dimensional strategy spaces used in the literature, we allow for a strategy space  $Q$  that is possibly infinite dimensional. In particular, we assume that  $Q$  is a compact complete separable metric space, i.e., a compact Polish space;

This paper is organized as follows. In section 2 we demonstrate how to proceed from a density model to a measure valued one and we formulate the model on the (natural) space of measures. In section 3 we establish the well-posedness of this model. In section 4 we demonstrate how this model encompasses the discrete, continuous replicator-mutator and species and quasi-species models. In section 5 we provide concluding remarks.

## 2 From Densities to Measures

We begin by giving a definition of a dynamical system that will be used throughout this paper.

**Definition 2.1.** If  $\mathfrak{T}, \Gamma$  are topological spaces, then a dynamical system on  $\mathfrak{T}$  is the tuple  $(\mathfrak{T}, \Gamma, \varphi)$  where,  $\varphi : \mathbb{R}_+ \times \mathfrak{T} \times \Gamma \rightarrow \mathfrak{T}$  is such that the following hold:

- i. For all  $(u, \gamma) \in \mathfrak{T} \times \Gamma$ ,  $\varphi(\cdot; u, \gamma)$  is continuous.
- ii. For all  $(u, \gamma) \in \mathfrak{T} \times \Gamma$ ,  $\varphi(0; u, \gamma) = u$ .
- iii. For all  $\theta_1, \theta_2, u, \gamma$ ,  $\varphi(\theta_1 + \theta_2; u, \gamma) = \varphi(\theta_2; \varphi(\theta_1, u, \gamma), \gamma)$ .
- iv. If  $\varphi$  is a continuous mapping then  $\varphi$  is called a continuous dynamical system.

There is a natural equivalence between dynamical systems and initial value problems. Given an initial value problem (IVP), the solution as a function of the parameter, initial condition and starting time generate a dynamical system [12]. Our dynamical system will be the one resulting from the solution of an IVP. To this end our initial modeling point is to take as the strategy space  $Q$  a compact subset of  $\text{int}(\mathbb{R}_+^n)$  (the interior of the positive cone of  $\mathbb{R}^n$ ). and to consider the following density IVP:

$$\begin{cases} \frac{d}{dt}x(t, q) = \underbrace{\int_Q f_1(X(t), \hat{q})p(q, \hat{q})x(t, \hat{q})d\hat{q}}_{\text{Birth term}} - \underbrace{f_2(X(t), q)x(t, q)}_{\text{Mortality term}} \\ x(0, q) = x_0. \end{cases} \quad (4)$$

Here,  $X(t) = \int_Q x(t, q)dq$  is the total population,  $f_1(X, \hat{q})$  represents the density-dependent replication rate per  $\hat{q}$  individual, while  $f_2(X, q)$  represents the density-dependent mortality rate per  $q$  individual. The probability density function  $p(q, \hat{q})$  is the selection-mutation kernel. That is,  $p(q, \hat{q})dq$  represents the probability that an individual of type  $\hat{q}$  replicates an individual of type  $q$  or the proportion of  $\hat{q}$ 's offspring that belong to the  $dq$  ball. Hence,  $f_1(X(t), \hat{q})p(q, \hat{q})dq$  is the offspring of  $\hat{q}$  in the  $dq$  ball and  $f_1(X(t), \hat{q})p(q, \hat{q})dqx(t, \hat{q})d\hat{q}$  is the total replication of the  $d\hat{q}$  ball into the  $dq$  ball. Summing (integrating) over all  $d\hat{q}$  balls results in the replication term. Clearly  $f_2(X(t), q)x(t, q)dq$  represents the mortality in the  $dq$  ball. The difference between birth and death in the  $dq$  ball gives the net rate of change of the individuals in the  $dq$  ball, i.e.,  $\frac{d}{dt}x(t, q)dq$ . Dividing by  $dq$  we get (4).

We point out that **formally**, if we let  $p(q, \hat{q}) = \delta_{\hat{q}}(q) = \delta_q(\hat{q})$  (the delta function is even) in (4) then we obtain the following pure selection (density) model

$$\begin{cases} \frac{d}{dt}x(t, q) = x(t, q)(f_1(X(t), q) - f_2(X(t), q)) \\ x(0, q) = x_0, \end{cases} \quad (5)$$

of which equation (1) in [2] is a special case. Indeed if  $p(q, \hat{q})dq = dq\delta_{\hat{q}}(q)$  then this means that the proportion of  $\hat{q}$ 's offspring in the  $dq$  ball is zero unless  $q = \hat{q}$  in which case this proportion is  $dq$ , i.e., individuals of type  $\hat{q}$  only give birth to individuals of type  $\hat{q}$ .

Integrating both sides of (4) over a Borel set  $E \subset Q$ , we obtain

$$\int_E \frac{d}{dt} x(t, q) dq = \int_E \left[ \int_Q f_1(X(t), \hat{q}) p(q, \hat{q}) x(t, \hat{q}) d\hat{q} - f_2(X(t), q) x(t, q) \right] dq.$$

Changing order of integration we get

$$\begin{aligned} \int_E \frac{d}{dt} x(t, q) dq &= \int_Q f_1(X(t), \hat{q}) \left[ \int_E p(q, \hat{q}) dq \right] x(t, \hat{q}) d\hat{q} - \int_E f_2(X(t), q) x(t, q) dq \\ &= \int_Q f_1(X(t), \hat{q}) \gamma(\hat{q})(E) x(t, \hat{q}) d\hat{q} - \int_E f_2(X(t), q) x(t, q) dq, \end{aligned}$$

where  $\gamma(\hat{q})(E) = \int_E p(q, \hat{q}) dq$  is the proportion of  $\hat{q}$ 's offspring in the Borel set  $E$ .

This yields the following measure valued dynamical system:

$$\begin{cases} \frac{d}{dt} \mu(t; u, \gamma)(E) = \int_Q f_1(\mu(t)(Q), \hat{q}) \gamma(\hat{q})(E) d\mu(t)(\hat{q}) \\ \quad - \int_E f_2(\mu(t)(Q), q) d\mu(t)(q) = F(\mu, \gamma)(E) \\ \mu(0; u, \gamma) = u. \end{cases} \quad (6)$$

### 3 Well-Posedness of Measure-Valued Dynamics

In this section we focus on the well-posedness of the model (6). This requires setting up some notation and notions and establishing several lemmas and propositions. To this end, throughout Section 3 the strategy space  $(Q, d)$  will be a compact complete separable metric space otherwise known as a compact Polish space. The reader may think of a compact Riemannian manifold or a compact subset of  $\text{int}(\mathbb{R}_+^n)$ .

#### 3.1 Birth and Mortality Rates

Concerning the birth and mortality densities  $f_1$  and  $f_2$  we make assumptions similar to those used in [2]:

- (A1)  $f_1 : \mathbb{R}_+ \times Q \rightarrow \mathbb{R}_+$  is locally Lipschitz continuous in  $X$  uniformly with respect to  $q$ , nonnegative, and nonincreasing on  $\mathbb{R}_+$  in  $X$  and continuous in  $q$ .

(A2)  $f_2 : \mathbb{R}_+ \times Q \rightarrow \mathbb{R}_+$  is locally Lipschitz continuous in  $X$  uniformly with respect to  $q$ , nonnegative, nondecreasing on  $\mathbb{R}_+$  in  $X$ , continuous in  $q$  and  $\inf_{q \in Q} f_2(0, q) = \varpi > 0$ .  
(This means that there is some inherent mortality not density related)

These assumptions are of sufficient generality to capture many nonlinearities of classical population dynamics including Ricker, Beverton-Holt, and Logistic (e.g., see [2]).

## 3.2 Technical Preliminaries for Measure Valued Formulation

### 3.2.1 Important Notation and Technical Definitions

We will use the symbol  $\mathcal{M}$  to denote the set of finite signed Borel measures when we wish to view it as a Riesz space [5] and  $\mathcal{M}_+$  will denote its positive cone. If the total variation norm is denoted  $|\cdot|_V$ , then  $\mathcal{M}_V$  will denote the Banach space of the finite signed measures with the total variation norm. Definition 6.2 in the Appendix tells us that the duality  $\langle C(Q), \mathcal{M} \rangle$  given by  $\langle f, \mu \rangle \mapsto \int_Q f(q) d\mu$  generates a *weak\** topology on  $\mathcal{M}$  which we denote as  $\mathcal{M}_w$ , i.e., the locally convex TVS (topological vector space)  $(\mathcal{M}, \sigma(\mathcal{M}, C(Q)))$ . If  $S \subseteq \mathcal{M}$ ,  $S_w$  denotes the same set under the *weak\** topology and  $S_V$  the same set under total variation. If no topology is indicated then  $S$  is simply a subset of the Riesz space of ordered measures. Also  $S_+ = S \cap \mathcal{M}_+$ . Let  $\mathcal{P}_w$  denote the probability measures under the *weak\** topology and  $C^{po} = C(Q, \mathcal{P}_w(Q))$ , the continuous functions on  $Q$  with the topology of uniform convergence.

Note that the EGT model we study here is a dynamical system arising from an ODE. A common method used to establish existence and uniqueness of solutions to such dynamical systems is to apply a contraction mapping argument to a suitably chosen complete metric space. Indeed, this is the method we adopt here.

To this end if  $a, b > 0$  and  $\mu_0 \in \mathcal{M}_+$  are given, let  $I_b(0)$  be the interval  $[0, b)$ , and  $\overline{B_a(\mu_0)}$  be the closed total variation ball of radius  $a$  around  $\mu_0$ . Since the space of finite signed measures  $\mathcal{M}_V$  under total variation norm is a Banach Space, if  $\mathcal{X}$  is any set then the bounded maps from  $\mathcal{X}$  into  $\mathcal{M}_V$  under the sup norm, i.e.,  $\|f\|_S = \sup_{x \in \mathcal{X}} |f(x)|_V$  is another Banach space denoted  $\mathcal{BM}(\mathcal{X}) := (\mathcal{BM}(\mathcal{X}), \|\cdot\|_S)$ .  $\mathcal{BM}(\mathcal{X})$  is the space in which we are *always* working and should be kept in mind when we begin the fixed point argument as there are several topologies being used. For our dynamical system purposes we are interested in the set  $\mathcal{X} = \overline{I_b(0)} \times (\overline{B_{a,+}(\mu_0)})_w \times C^{po}$ . Let's denote by  $\mathcal{C}(\overline{I_b(0)} \times (\overline{B_{a,+}(\mu_0)})_w \times C^{po}; (\overline{B_{2a}(\mu_0)})_w)$  the closed subcollection of continuous maps into  $(\overline{B_{2a}(\mu_0)})_w$ . Then it is an exercise to show

that  $(M(a, b), \|\cdot\|_S)$  where

$$M(a, b) = \{\alpha \in \mathcal{BM}(\mathcal{X}) \mid \alpha \in \mathcal{C}(\overline{I_b(0)} \times \overline{(B_{a,+}(\mu_0))_w} \times C^{po}; \overline{(B_{2a}(\mu_0))_w}), \\ \alpha \geq 0, \alpha(0; u, \gamma) = u\}$$

is a nonempty closed metric subspace of the complete metric space  $\mathcal{BM}(\overline{I_b(0)} \times \overline{B_{a,+}(\mu_0)} \times C^{po})$ .

We will let  $\vec{0}$  denote the zero measure,  $\mathbf{1}$  denote the constant function one (from  $Q$  to  $\mathbb{R}$ ), and if  $\alpha \in M(a, b)$ , then we will at times write  $\alpha(t)$  for  $\alpha(t; u, \gamma)$  when we are keeping  $u, \gamma$  fixed.

**3.2.2 Families of Measures and Mutation Kernels**  $\left( \int_Q f_1(X, \hat{q}) \gamma(\hat{q}) d\mu(\hat{q}), \right.$

$$\left. \int_{T \times Q} f_1(X(s), \hat{q}) \overline{\gamma}_{s,t,\alpha(\cdot; u, \gamma)}(\hat{q}) d\mu(\hat{q}) \times ds \right)$$

In order to understand this section we must first understand all of the duals that we will be using. As they can be confusing. Given a vector space  $V$  or more generally a Riesz space, one automatically has an algebraic dual denoted  $V^\sharp$ . If  $V$  is also a topological vector space, then there is the continuous dual denoted  $V'$  with the relation  $V' \subseteq V^\sharp$ . Since  $V'$  is also a vector space we can form  $V'^\sharp$  which has the relation  $V \subseteq V'^\sharp \subseteq V^\sharp$ . The first  $\subseteq$  is actually the natural algebraic monomorphism  $v \mapsto \delta_v$ . So given  $v \in V$  there are three ways to view this element given by each inclusion. We shall have occasion to use this fact when defining our mutation term.

A measure is both a countably additive set function and also a continuous linear functional on  $C(Q)$  [7]. For example, if  $\nu$  is a measure

$$\nu(\mathbf{1}) = \nu(Q) = \int_Q d\nu.$$

Each view is useful in its own right. For example, if one wishes to model the sizes of populations then speaking of the “measure” of a Borel set intuitively has the meaning size of population. Speaking of the value of a linear functional on a continuous function is less intuitive biologically. However, for mathematical purposes at times the linear functional viewpoint is more beneficial. So in our proofs we will use the functional definition, however in biological explanations we will use the set function approach.



We are all familiar with point masses and absolutely continuous measures. However, in the formulation of this model we come upon a novel type of measure. This measure is defined as the integral of a family of measures. If  $T$  is a closed interval of  $\mathbb{R}_+$ , then  $T \times Q$  is compact. If  $\gamma \in C^{po}$ ,  $\alpha \in M(a, b)$ , then define

$$\overline{\gamma}_{s,t,\alpha(\cdot;u,\gamma)}(\hat{q})(E) = \int_E e^{-\int_s^t f_2(\alpha(\tau)(Q),q)d\tau} d\gamma(\hat{q})(q).$$

From a biological point of view  $\overline{\gamma}_{s,t,\alpha(\cdot;u,\gamma)}(\hat{q})(E)$  is the net proportion of  $\hat{q}$ 's offspring that belong to  $E$  from time  $s$  to time  $t$ . Since  $f_1(\varphi(s;u,\gamma)(Q),\hat{q})\mu(d\hat{q})$  is the number of offspring produced by a  $d\hat{q}$  ball,  $f_1(\varphi(s;u,\gamma)(Q),\hat{q})\overline{\gamma}_{s,t,\alpha(\cdot;u,\gamma)}(\hat{q})(E)\mu(d\hat{q})$  is the total contribution of the  $d\hat{q}$  ball to the Borel set  $E$  by total new recruits from time  $s$  to  $t$ .

If  $f_1$  is bounded and  $\gamma \in C^{po}$ , then we wish to consider two mappings: (1) for each  $X$  the mapping  $\hat{q} \mapsto f_1(X,\hat{q})\gamma(\hat{q})$ ; (2)  $(s,\hat{q}) \mapsto f_1(X(s),\hat{q})\overline{\gamma}_{s,t,\alpha(\cdot;u,\gamma)}(\hat{q})$ . They are both weakly continuous mappings with compact support that map into a complete convex subset of the locally convex space  $\mathcal{M}_w$ . So if  $\mu \in \mathcal{M}$ , then  $\int_Q f_1(X,\hat{q})\gamma(\hat{q})d\mu(\hat{q})$  and  $\int_{T \times Q} f_1(X(s),\hat{q})\overline{\gamma}_{s,t,\alpha(\cdot;u,\gamma)}(\hat{q})(d\mu(\hat{q}) \times ds)$  exists and are also elements of  $\mathcal{M}$  by Theorem 6.7 in the Appendix. Let us be more clear. These two integrals are elements of  $\mathcal{M}$  in the following sense. Let  $\hat{\cdot}$  denote the canonical algebraic imbedding  $\hat{\cdot}: \mathcal{M} \hookrightarrow (\mathcal{M}^\#)^\#$  given by  $\hat{\nu}(f) = f(\nu)$  or  $\hat{\nu} = \delta_\nu$ , where  $\delta_\nu(f) = f(\nu)$  for  $f \in \mathcal{M}^\#$  is the evaluation homomorphism. Since  $(\mathcal{M}_w)' \subseteq \mathcal{M}^\#$ ,  $(\mathcal{M}^\#)^\# \subseteq (\mathcal{M}'_w)^\#$ . So  $\nu \mapsto \delta_\nu$  actually algebraically imbeds  $\mathcal{M} \hookrightarrow (\mathcal{M}'_w)^\#$ . So viewing  $\nu$  as the algebraic linear functional  $\delta_\nu$  is what we mean. More to the point, let us consider only the first integral  $\int_Q f_1(X,\hat{q})\gamma(\hat{q})d\mu(\hat{q})$ , since the second can be understood similarly. By Theorem 6.7 and the above discussion  $\int_Q f_1(X,\hat{q})\gamma(\hat{q})d\mu(\hat{q})$  is an element  $\delta_\nu \in (\mathcal{M}'_w)^\#$  and by Definition 6.5

$$z'(\nu) = \delta_\nu(z') = \mu(z'(f_1\gamma)) \quad \text{where} \quad z' \in (\mathcal{M}_w)'.$$

Since  $\langle C(Q), \mathcal{M}_V \rangle$  is a duality by Theorem 6.3, for  $z' \in (\mathcal{M}_w)'$  there is a unique  $z \in C(Q)$  such that  $z'(\mu) = \langle z, \mu \rangle$  for all  $\mu \in \mathcal{M}$ . Hence,

$$\langle z', \hat{\nu} \rangle = \langle z', \delta_\nu \rangle = \delta_\nu(z') = z'(\nu) = \langle z, \nu \rangle, \quad \text{for all } z' \in (\mathcal{M}_w)'. \quad (7)$$

So if  $\int_Q f_1(X,\hat{q})\gamma(\hat{q})d\mu(\hat{q}) = \delta_\nu = \hat{\nu}$ , then we define  $\int_Q f_1(X,\hat{q})\gamma(\hat{q})d\mu(\hat{q})$  to be the measure  $\nu$  which behaves as in (7). Similarly for  $\int_{[0,T] \times Q} f_1(X(s),\hat{q})\overline{\gamma}_{s,t,\alpha(\cdot;u,\gamma)}(\hat{q})(d\mu(\hat{q}) \times ds)$ .

If  $f$  is continuous, the measure  $E \mapsto \int_E f(q) d\mu(q)$  as a functional has the action:  $z \mapsto \int_Q z(q) f(q) d\mu(q)$  for  $z \in C(Q)$ . For the remainder of this section we will denote such a functional as  $\langle \int f(q) d\mu(q), \cdot \rangle$ . Before we end this section we will draw a connection between the continuous functional and set function aspects of these families of measures.

**Theorem 3.1.** *Let  $\mu \in \mathcal{M}_+$ . If  $f : Q \rightarrow \mathcal{M}_w$  is continuous and bounded in total variation, then*

$$\left( \int_Q f(\hat{q}) d\mu \right)(E) = \int_Q f(\hat{q})(E) d\mu$$

for every Borel set  $E$ .

*Proof.* We give a sketch of the proof and refer the reader to [5, 7, 29] for background definitions and details. Since  $Q$  is a metric space, it is outer normal, hence outer regular [5, pg. 379]. Thus, the value of a finite signed measure is completely known once it is known on open sets. To this end let  $\nu_2(E) = \int_Q f(\hat{q})(E) d\mu$ . Then it is an elementary exercise to demonstrate that  $\nu_2$  is a finite signed measure [29]. Using Theorem 6.7 and the analysis before this theorem  $\nu_1 = \int_Q f(\hat{q}) d\mu \in \mathcal{M}$ .

We will show that  $\nu_1 = \nu_2$  on open sets. By definition, since the characteristic functions of open sets are lower semi-continuous, if  $G$  is open and  $\varphi_G$  is its characteristic function, then

$$\begin{aligned} \nu_1(G) &:= \nu_1^*(\varphi_G) = \sup_{h \in C(Q), h \leq \varphi_G} \nu_1(h) = \sup_{h \in C(Q), h \leq \varphi_G} \int_Q f(\hat{q})(h) d\mu \\ &= \int_Q \sup_{h \in C(Q), h \leq \varphi_G} f(\hat{q})(h) d\mu = \int_Q (f(\hat{q}))^*(\varphi_G) d\mu =: \nu_2(G). \end{aligned}$$

□

### 3.3 Main Well-Posedness Theorem

The following is the main theorem of this section.

**Theorem 3.2.** *There exists a continuous dynamical system  $(\mathcal{M}_{+,w}, C^{po}, \varphi)$  where  $\varphi : \mathbb{R}_+ \times \mathcal{M}_{+,w} \times C^{po} \rightarrow \mathcal{M}_{+,w}$  satisfies the following:*

1. *For fixed  $u, \gamma$ , the mapping  $t \mapsto \varphi(t; u, \gamma)$  is continuously differentiable in total variation, i.e.,  $\varphi(\cdot, u, \gamma) : \mathbb{R}_+ \rightarrow \mathcal{M}_{V,+}$ .*

2. For fixed  $u, \gamma$ , the mapping  $t \mapsto \varphi(t; u, \gamma)$  is the unique solution to

$$\begin{cases} \frac{d}{dt}\mu(t)(E) = \int_Q f_1(\mu(t)(Q), \hat{q})\gamma(\hat{q})(E)d\mu(t)(\hat{q}) \\ \quad - \int_E f_2(\mu(t)(Q), \hat{q})d\mu(t)(\hat{q}) = F(\mu, \gamma)(E) \\ \mu(0) = u. \end{cases} \quad (8)$$

We now establish a few results that are needed to prove Theorem 3.2.

### 3.3.1 Local Existence and Uniqueness of Dynamical System

First let  $\mu_0 \in \mathcal{M}_+$  and  $a > 0$  be fixed. As it stands  $F(\mu, \gamma)$  as defined in (8) need not be a finite signed measure at all. If  $\mu(t)(Q)$  is ever negative, then  $F(\mu(t), \gamma)$  is not defined. So we modify  $F$  as follows: Choose  $\tilde{K} > \mu_0(Q) + 2a$ . For  $j = 1, 2$ , extend  $f_j$  to  $\mathbb{R} \times Q$  by setting  $\tilde{f}_j(x, q) = f_j(0, q)$  for  $x \leq 0$  and make the modification  $\tilde{f}_j(x, q) = f_j(\tilde{K}, q)$  for  $x \geq \tilde{K}$ . Then  $\tilde{f}_j : \mathbb{R} \times Q \rightarrow \mathbb{R}_+$  are Lipschitz continuous in the first variable and bounded with Lipschitz constants  $L_j$  and bounds  $B_j$ . Let  $\tilde{F}(\mu, \gamma)(E)$  be the redefined vector field obtained by replacing  $f_j$  with  $\tilde{f}_j$ . The function  $\tilde{F}(\mu, \gamma)$  is now a finite signed measure.

**Lemma 3.3.** (*Lipschitz  $F$* ) Let  $\tilde{F}$  be as above and let  $W \subseteq \mathcal{M}$  be bounded in total variation. Then for every  $\gamma \in C^{po}$  we have the following:

1. There exists a continuous function  $K_{\tilde{F}} \geq 0$ , such that  $|\tilde{F}(\alpha, \gamma)|_V \leq K_{\tilde{F}}(|\alpha|_V)|\alpha|_V$ ,  $\forall \alpha \in \mathcal{M}$ .
2.  $\tilde{F}(\alpha, \gamma)$  is bounded and uniformly Lipschitz continuous on  $(W_+)_V \times C^{po}$  in  $\alpha$ .

*Proof.* 1. Define  $K_{\tilde{F}}(s) = B_1 + B_2 + (L_1 + L_2)s$ . Since  $F(\vec{0}, \gamma) = \vec{0}$ , this follows from equation (9) below.

2. Let  $C_W$  be a bound for  $W$  in the norm topology, i.e.,  $|\mu|_V \leq C_W$  for  $\mu \in W$ . We now prove uniform Lipschitz continuity in  $\alpha$ . The boundedness trivially follows. Given  $W$ , notice that for all  $\alpha \in W$ ,  $K_{\tilde{F}}(|\alpha|_V) \leq K_{\tilde{F}}(C_W)$ . If  $\alpha$  and  $\beta$  are finite signed measures,

then  $d(\alpha) = d(\alpha - \beta + \beta)$ . Hence,

$$\begin{aligned}\tilde{F}(\alpha, \gamma) - \tilde{F}(\beta, \gamma) &= \int_Q \gamma(\hat{q}) [\tilde{f}_1(\alpha(Q), \hat{q}) - \tilde{f}_1(\beta(Q), \hat{q})] d\alpha(\hat{q}) \\ &\quad + \int_Q \tilde{f}_1(\beta(Q), \hat{q}) \gamma(\hat{q}) d(\alpha - \beta)(\hat{q}) \\ &\quad - < \int [\tilde{f}_2(\alpha(Q), \hat{q}) - \tilde{f}_2(\beta(Q), \hat{q})] d\alpha(\hat{q}), \cdot > \\ &\quad - < \int \tilde{f}_2(\beta(Q), \hat{q}) d(\alpha - \beta)(\hat{q}), \cdot >, \end{aligned}$$

and

$$|\tilde{F}(\alpha, \gamma) - \tilde{F}(\beta, \gamma)|_V \leq |\alpha|_V L_1 |\alpha - \beta|_V + B_1 |\alpha - \beta|_V + |\alpha|_V L_2 |\alpha - \beta|_V + B_2 |\alpha - \beta|_V.$$

Thus,

$$|\tilde{F}(\alpha, \gamma) - \tilde{F}(\beta, \gamma)|_V \leq K_{\tilde{F}}(|\alpha|_V) |\alpha - \beta|_V \leq K_{\tilde{F}}(C_W) |\alpha - \beta|_V. \quad (9)$$

□

**Lemma 3.4.** (Estimates) If  $\alpha, \beta \in M(a, b)$ ,  $t_1, t_2 \in \mathbb{R}_+$ ,  $\mu_0 \in \mathcal{M}_+$  pick constants  $C_1, C_2$  as follows:  $C_1 = \mu_0(Q) + 2a$ ,  $C_2 = L_1 + 2bL_2B_1$ . We have the following estimates:

1.  $\left| \int_{[t_1, t_2] \times Q} \tilde{f}_1(\alpha(s)(Q), \hat{q}) \overline{\gamma}_{s, t, \alpha}(\hat{q}) d\alpha(s) ds \right|_V \leq 2bC_1B_1.$
2.  $|e^{-\int_{t_1}^{t_2} \tilde{f}_2(\alpha(\tau)(Q), q) d\tau} - e^{-\int_{t_1}^{t_2} \tilde{f}_2(\beta(\tau)(Q), q) d\tau}| \leq \|(\alpha - \beta)\|_S L_2 2b$  for all  $t_1, t_2 \in I_b(0)$ .
3.  $\left| \tilde{f}_1(\alpha(s; u, \gamma)(Q), \hat{q}) \overline{\gamma}_{s, t, \alpha}(\hat{q}) - \tilde{f}_1(\beta(s; u, \gamma)(Q), \hat{q}) \overline{\gamma}_{s, t, \beta}(\hat{q}) \right|_V \leq C_2 \|\alpha - \beta\|_S.$

*Proof.* 1. Recall from [5, pg. 185] that  $\|\nu\|_V = \sup_{\|f\|_\infty \leq 1} |< \nu, f >|$ . Initially if  $f \in C(Q)$ ,

then we have by using Definition 6.5

$$\begin{aligned} &\left| \int_{[t_1, t_2] \times Q} \tilde{f}_1(\alpha(s)(Q), \hat{q}) \overline{\gamma}_{s, t, \alpha}(\hat{q}) d\alpha(s) ds \right|_V \\ &= \sup_{\|f\|_\infty \leq 1} |< \int_{[t_1, t_2] \times Q} \tilde{f}_1 \overline{\gamma}_{s, t, \alpha}, f >| \\ &= \sup_{\|f\|_\infty \leq 1} |< \left( \int_{[t_1, t_2] \times Q} \tilde{f}_1 \overline{\gamma}_{s, t, \alpha} \right), f' >| \\ &= \sup_{\|f\|_\infty \leq 1} \left| \int_{[t_1, t_2] \times Q} < \tilde{f}_1 \overline{\gamma}_{s, t, \alpha}(\hat{q}), f' > d\alpha(s) \times ds \right| \\ &\leq 2bC_1 \sup_{\|f\|_\infty \leq 1, (s, \hat{q}) \in [t_1, t_2] \times Q} |< \tilde{f}_1 \overline{\gamma}_{s, t, \alpha}(\hat{q}), f' >| \leq 2bC_1B_1, \end{aligned}$$

since

$$\begin{aligned} | \langle \tilde{f}_1 \bar{\gamma}_{s,t,\alpha}(\hat{q}), f' \rangle | &= \left| \int_Q f(q) d(\tilde{f}_1 \bar{\gamma}_{s,t,\alpha}(\hat{q}))(q) \right| \\ &= \left| \int_Q f(q) \tilde{f}_1(\alpha(s)(Q), \hat{q}) e^{-\int_s^t \tilde{f}_2(\alpha(\tau)(Q), q) d\tau} d\gamma(\hat{q})(q) \right| \leq B_1. \end{aligned}$$

(see subsection 3.2.2 for the notation  $\widehat{\left( \int_{[t_1, t_2] \times Q} \tilde{f}_1 \bar{\gamma}_{s,t,\alpha} \right)}$  and  $f'$ ).

2. There exists  $\xi > 0$ , such that

$$\begin{aligned} &\left| e^{-\int_{t_1}^{t_2} \tilde{f}_2(\alpha(\tau)(Q), q) d\tau} - e^{-\int_{t_1}^{t_2} \tilde{f}_2(\beta(\tau)(Q), q) d\tau} \right| \\ &= e^{-\xi} \left| \int_{t_1}^{t_2} \left[ \tilde{f}_2(\beta(\tau; u, \gamma)(Q), q) - \tilde{f}_2(\alpha(\tau; u, \gamma)(Q), q) \right] d\tau \right| \\ &\leq 2bL_2 \|\alpha - \beta\|_S. \end{aligned}$$

3. For the third estimate we have:

$$\begin{aligned} &\left| \tilde{f}_1(\alpha(s; u, \gamma)(Q), \hat{q}) \bar{\gamma}_{s,t,\alpha}(\hat{q}) - \tilde{f}_1(\beta(s; u, \gamma)(Q), \hat{q}) \bar{\gamma}_{s,t,\beta}(\hat{q}) \right|_V \\ &\leq \left| \tilde{f}_1(\alpha(s; u, \gamma)(Q), \hat{q}) - \tilde{f}_1(\beta(s; u, \gamma)(Q), \hat{q}) \right| \left| \bar{\gamma}_{s,t,\alpha}(\hat{q}) \right|_V \\ &\quad + \left| \tilde{f}_1(\beta(s; u, \gamma)(Q), \hat{q}) \right| \left| \bar{\gamma}_{s,t,\alpha}(\hat{q}) - \bar{\gamma}_{s,t,\beta}(\hat{q}) \right|_V \\ &\leq L_1 |\alpha(s) - \beta(s)|(Q) + 2bL_2 B_1 \|\alpha - \beta\|_S \leq (L_1 + 2bB_1 L_2) \|\alpha - \beta\|_S. \end{aligned}$$

□

**Lemma 3.5.** (*Fixed Point*) If  $\mu_0 \in \mathcal{M}_+$ , let  $a > 0$ ,  $C_1, C_2$  be as in Lemma 3.4, with  $b$  such that  $(1 - e^{-B_2 b})\mu_0(Q) + 2B_1 C_1 b < a$  and  $b < \min\{1, \frac{1}{2L_2 C_1 + 2B_1 + 2C_2 C_1}\}$ . Then  $S: M(a, b) \rightarrow M(a, b)$  given by

$$\begin{aligned} [S\alpha](t; u, \gamma) &= \left\langle \int_0^t e^{-\int_0^s \tilde{f}_2(\alpha(\tau)(Q), q) d\tau} du(q), \cdot \right\rangle \\ &\quad + \left( \int_0^t \int_Q \tilde{f}_1(\alpha(s)(Q), \hat{q}) \bar{\gamma}_{s,t,\alpha}(\hat{q}) d\alpha(s)(\hat{q}) ds \right) \end{aligned} \tag{10}$$

has a unique fixed point.

*Proof.* Let  $\alpha \in M(a, b)$ . Now clearly from the form of (10)  $[S\alpha](0, u, \gamma) = u$  and  $[S\alpha]$  is nonnegative. If  $a, b, C_1, C_2$  are as in the hypothesis, then

$$\begin{aligned} ([S\alpha](t, u, \gamma) - \mu_0) &= < \int e^{-\int_0^t \tilde{f}_2(\alpha(\tau)(Q), q) d\tau} d(u - \mu_0), \cdot > \\ &+ < \int (e^{-\int_0^t \tilde{f}_2(\alpha(\tau)(Q), q) d\tau} - 1) d\mu_0(q), \cdot > \\ &+ \left( \int_0^t \int_Q \tilde{f}_1(\alpha(s)(Q), \hat{q}) \bar{\gamma}_{s,t,\alpha}(\hat{q}) d\alpha(s)(\hat{q}) \times ds \right) \end{aligned}$$

and

$$\begin{aligned} |[S\alpha] - \mu_0|_V &\leq |u - \mu_0|_V + (1 - e^{-B_2 b})\mu_0(Q) + 2bB_1C_1 \\ &\leq a + (1 - e^{-B_2 b})\mu_0(Q) + 2bB_1C_1 < 2a. \end{aligned}$$

We now show that  $[S\alpha]$  is continuous. This means that if  $(t_n, u_n, \gamma_n)$  is a sequence in  $\overline{I_b(0)} \times (\overline{B_{a,+}(\mu_0)})_w \times C^{po}$  that converges to  $(t, u, \gamma) \in \overline{I_b(0)} \times (\overline{B_{a,+}(\mu_0)})_w \times C^{po}$ , and if  $[S\alpha]_n = [S\alpha](t_n; u_n, \gamma_n)$  and  $[S\alpha] = [S\alpha](t; u, \gamma)$ , then  $[S\alpha]_n \rightarrow [S\alpha]$  in the weak\* topology. Let

$$\begin{aligned} Ia &= < \int e^{-\int_0^{t_n} \tilde{f}_2(\alpha_n(\tau)(Q), q) d\tau} d(u_n - u)(q), \cdot >, \\ Ib &= < \int [e^{-\int_0^{t_n} \tilde{f}_2(\alpha_n(\tau)(Q), q) d\tau} - e^{-\int_0^t \tilde{f}_2(\alpha(\tau)(Q), q) d\tau}] du, \cdot >, \\ IIa &= \left( \int_t^{t_n} \int_Q \tilde{f}_1(\alpha_n(s)(Q), \hat{q}) \bar{\gamma}_{s,t_n,\alpha_n}(\hat{q}) d\alpha_n(\hat{q}) ds \right), \\ IIb1 &= \left( \int_0^t \int_Q [\tilde{f}_1(\alpha_n(s)(Q), \hat{q}) - \tilde{f}_1(\alpha(s)(Q), \hat{q})] \bar{\gamma}_{s,t_n,\alpha_n}(\hat{q}) d\alpha_n(\hat{q}) ds \right), \\ IIb2 &= \left( \int_0^t \int_Q \tilde{f}_1(\alpha(s)(Q), \hat{q}) [\bar{\gamma}_{s,t_n,\alpha_n}(\hat{q}) - \bar{\gamma}_{s,t,\alpha}(\hat{q})] d\alpha_n(\hat{q}) ds \right), \text{ and} \\ IIb3 &= \left( \int_0^t \int_Q \tilde{f}_1(\alpha(s)(Q), \hat{q}) \bar{\gamma}_{s,t,\alpha}(\hat{q}) d[\alpha_n - \alpha](\hat{q}) ds \right). \end{aligned}$$

Then,  $([S\alpha]_n - [S\alpha]) = Ia + Ib + IIa + IIb1 + IIb2 + IIb3$ . We remind the reader that the weak\* topology is generated the family of seminorms  $\rho_f(\mu) = |\int_Q f d\mu|$ , where  $f \in C(Q)$ .

So if  $\rho_f$  is a seminorm, we need to show that  $\rho_f([S\alpha]_n - [S\alpha])$  is small as  $n \rightarrow \infty$ . To this end, we provide an estimate for each of the terms above.

1.  $\rho_f(Ia)$  is small since  $e^{-\int_0^{t_n} \tilde{f}_2(\alpha_n(\tau)(Q), q) d\tau} \rightarrow e^{-\int_0^t \tilde{f}_2(\alpha(\tau)(Q), q) d\tau}$  uniformly in  $q$ ,  $e^{-\int_0^t \tilde{f}_2(\alpha(\tau)(Q), q) d\tau}$  is continuous in  $q$  and  $u_n \rightarrow u$  in  $\mathcal{M}_w$ .
2. The fact that  $\rho_f(Ib)$  is small follows from the fact that  $e^{-\int_0^{t_n} \tilde{f}_2(\alpha_n(\tau)(Q), q) d\tau} \rightarrow e^{-\int_0^t \tilde{f}_2(\alpha(\tau)(Q), q) d\tau}$  uniformly in  $q$ .

3. The fact that  $\rho_f(IIb1)$  is small follows from the second estimate in Lemma 3.4 and Theorem 6.6.
4. Using Theorem 6.6 we get

$$\rho_f(IIb2) \leq \int_0^t \int_Q \tilde{f}_1(\alpha(s)(Q), \hat{q}) \rho_f[\bar{\gamma}_{s,t_n,\alpha_n}(\hat{q}) - \bar{\gamma}_{s,t,\alpha}(\hat{q})] d|\alpha_n|(\hat{q}) ds.$$

Since  $e^{-\int_s^{t_n} \tilde{f}_2(\alpha_n(\tau)(Q), q) d\tau} \rightarrow e^{-\int_s^t \tilde{f}_2(\alpha(\tau)(Q), q) d\tau}$  uniformly in  $(s, q)$ , then  $\rho_f[\bar{\gamma}_{s,t_n,\alpha_n}(\hat{q}) - \bar{\gamma}_{s,t,\alpha}(\hat{q})] \rightarrow 0$  uniformly in  $(s, \hat{q})$  as  $n \rightarrow \infty$ . Thus, our result is immediate.

5. For the term  $IIb3$  we have

$$\begin{aligned} \rho_f(IIb3) = \\ \left| \int_0^t \int_Q \tilde{f}_1(\alpha(s)(Q), \hat{q}) \int_Q f(q) e^{-\int_s^t \tilde{f}_2(\alpha(\tau)(Q), q) d\tau} d\gamma(\hat{q})(q) d[\alpha_n - \alpha](\hat{q}) ds \right|. \end{aligned}$$

If  $g_n(s) = \int_Q \tilde{f}_1(\alpha(s)(Q), \hat{q}) \int_Q f(q) e^{-\int_s^t \tilde{f}_2(\alpha(\tau)(Q), q) d\tau} d\gamma(\hat{q})(q) d[\alpha_n - \alpha](\hat{q})$ , then  $g_n \rightarrow 0$  pointwise. Hence our result follows by dominated convergence and the facts that  $\tilde{f}_1(\alpha(s)(Q), \hat{q}) \int_Q f(q) e^{-\int_s^t \tilde{f}_2(\alpha(\tau)(Q), q) d\tau} d\gamma(\hat{q})(q)$  is continuous and  $\alpha_n \rightarrow \alpha$ .

6. By hypothesis  $t_n \rightarrow t$ , hence, the term  $\rho_f(IIa)$  is small since the integrands are bounded.

Now for the contraction we have the following. If

$$\begin{aligned} I &= \left\langle \int_0^t \left( e^{-\int_0^t \tilde{f}_2(\alpha(\tau)(Q), q) d\tau} - e^{-\int_0^t \tilde{f}_2(\beta(\tau)(Q), q) d\tau} \right) du(q), \cdot \right\rangle, \\ II &= \left( \int_0^t \int_Q \tilde{f}_1(\alpha(s)(Q), \hat{q}) \bar{\gamma}_{s,t,\alpha}(\hat{q}) d(\alpha - \beta)(s)(\hat{q}) ds \right), \\ III &= \left( \int_0^t \int_Q \left\{ \tilde{f}_1(\alpha(s)(Q), \hat{q}) \bar{\gamma}_{s,t,\alpha}(\hat{q}) - \tilde{f}_1(\beta(s)(Q), \hat{q}) \bar{\gamma}_{s,t,\beta}(\hat{q}) \right\} d\beta(s)(\hat{q}) ds \right), \end{aligned}$$

then  $([S\alpha] - [S\beta]) = I + II + III$ , and  $\|[S\alpha] - [S\beta]\|_V \leq |I|_V + |II|_V + |III|_V \leq (2bL_2C_1 + 2bB_1 + 2bC_1C_2)\|\alpha - \beta\|_S$ . Hence,  $S$  is a contraction mapping. Therefore,  $S$  has a unique fixed point in  $M(a, b)$ .  $\square$

We will denote this fixed point by  $\tilde{\varphi}_a$ .

**Proposition 3.6.** (Local Solution) *If  $\mu_0 \in \mathcal{M}_+$ , and  $b$  is as in Lemma 3.5, then*

1. the function  $\tilde{\varphi}_a$  satisfies

$$\begin{aligned} \tilde{\varphi}_a(t; u, \gamma) = & \int_0^t e^{-\int_0^t \tilde{f}_2(\tilde{\varphi}_a(\tau)(Q), q) d\tau} du(q), \cdot > \\ & + \left( \int_{[0, t] \times Q} \tilde{f}_1(\tilde{\varphi}_a(s; u, \gamma)(Q), \hat{q}) \overline{\gamma}_{s, t, \tilde{\varphi}} d\tilde{\varphi}_a(s)(\hat{q}) \times ds \right) \end{aligned} \quad (11)$$

and is a local solution to

$$\begin{cases} \dot{x}(t) = \tilde{F}(x(t), \gamma) \\ \quad = \left( \int_Q \tilde{f}_1(x(t)(Q), \hat{q}) \gamma(\hat{q}) dx(t)(\hat{q}) \right) - \int \tilde{f}_2(x(t)(Q), \hat{q}) dx(t)(\hat{q}), \cdot > \\ x(0) = u. \end{cases} \quad (12)$$

2.  $\tilde{\varphi}_a$  is nonnegative and continuous.

*Proof.* 1. We differentiate the integral representation (11) and show that it satisfies (12).

Then we use uniqueness of solution given that we have Lipschitzicity by Lemma 3.3.

If  $\tilde{\varphi}_a = \mu_1 + \mu_2$ , then  $\dot{\tilde{\varphi}}_a = \dot{\mu}_1 + \dot{\mu}_2$ , where

$$\mu_1 = \int_0^t e^{-\int_0^t \tilde{f}_2(\tilde{\varphi}_a(\tau)(Q), q) d\tau} du, \cdot >$$

and

$$\mu_2 = \int_{[0, t] \times Q} \tilde{f}_1(\tilde{\varphi}_a(s)(Q), \hat{q}) \overline{\gamma}_{s, t, \tilde{\varphi}} d\tilde{\varphi}_a(s)(\hat{q}) \times ds.$$

Clearly

$$\dot{\mu}_1 = \int -\tilde{f}_2(\tilde{\varphi}_a(t)(Q), q) d\mu_1(q), \cdot > .$$

Since

$$\mu_2(f) = \int_0^t \left[ \int_Q \tilde{f}_1(\tilde{\varphi}_a(s)(Q), \hat{q}) \int_Q f(q) e^{-\int_s^t \tilde{f}_2(\tilde{\varphi}_a(\tau)(Q), q) d\tau} d\gamma(\hat{q})(q) d\tilde{\varphi}_a \right] ds,$$

then

$$\begin{aligned} \dot{\mu}_2(f) &= \int_0^t \left[ \int_Q \tilde{f}_1(\tilde{\varphi}_a(s)(Q), \hat{q}) \int_Q f(q) (-\tilde{f}_2(\tilde{\varphi}_a(t)(Q), q)) d\gamma_{s, t, \tilde{\varphi}_a}(\hat{q})(q) d\tilde{\varphi}_a \right] ds \\ &\quad + \int_Q \tilde{f}_1(\tilde{\varphi}_a(t)(Q), \hat{q}) \int_Q f(q) d\gamma(\hat{q})(q) d\tilde{\varphi}_a(t)(\hat{q}) \\ &= \mu_2(-\tilde{f}_2(\tilde{\varphi}_a(t)(Q), \cdot) f) + \left( \int_Q \tilde{f}_1(\tilde{\varphi}_a(t)(Q), \hat{q}) \gamma(\hat{q}) d\tilde{\varphi}_a(t)(\hat{q}) \right) (f) \\ &= \int -\tilde{f}_2(\tilde{\varphi}_a(t)(Q), \cdot) d\mu_2, f > + \left( \int_Q \tilde{f}_1(\tilde{\varphi}_a(t)(Q), \hat{q}) \gamma(\hat{q}) d\tilde{\varphi}_a(t)(\hat{q}) \right) (f). \end{aligned}$$



Hence,

$$\dot{\tilde{\varphi}}_a = \left( \int_Q \tilde{f}_1(\tilde{\varphi}_a(t)(Q), \hat{q}) \gamma(\hat{q}) d\tilde{\varphi}_a(t)(\hat{q}) \right) - \left\langle \int \tilde{f}_2(\tilde{\varphi}_a(t)(Q), q) d\tilde{\varphi}_a(t), \cdot \right\rangle.$$

2. This follows from Lemma 3.5. □

For fixed  $u, \gamma$  we denote this local solution to (12) by  $\tilde{\mu}_a$ , i.e.,  $\tilde{\mu}_a(t) = \tilde{\varphi}_a(t; u, \gamma)$ . Since  $\tilde{\varphi}_a$  is nonnegative, we see that  $\tilde{\mu}_a$  is nonnegative.

### 3.3.2 Proof of Theorem 3.2

Let  $a > 0$ , by Proposition 3.6 we see that the dynamical system,  $\tilde{\varphi}_a$ , exists on a small interval  $I_b(0)$ . Since  $\tilde{\varphi}_a \in B_{2a}(\mu_0)$ , then  $\tilde{\varphi}_a(t; u, \gamma)(Q) < \tilde{K}$  and  $\tilde{F}(\tilde{\varphi}_a(t; u, \gamma), \gamma) = F(\tilde{\varphi}_a(t; u, \gamma), \gamma)$  on  $I_b(0)$ . Hence, equation (8) has the local solution  $\tilde{\varphi}_a$  on  $I_b(0)$ . This means by Lemma 3.6 that for fixed  $u$  and  $\gamma$ ,  $\tilde{\varphi}_a(\cdot, u, \gamma) : I_b(0) \rightarrow \mathcal{M}_{V,+}$  is continuously differentiable and satisfies (8). We will denote this solution as  $\mu_a$  and the dynamical system as  $\varphi_a$ .

Moreover, from the nonnegativity of the local solution to (8),  $\mu_a$ , and the nonincreasing property of  $f_1$  with respect to  $X$  given in assumption (A1), it is easy to show that this solution satisfies  $\dot{\mu}_a(t)(Q) \leq M_{f_1} \mu_a(t)(Q)$ , where  $M_{f_1} = \max_{q \in Q} f_1(0, q)$ . Hence, if we let  $g(t, s) = M_{f_1} s$ , then using Theorem 6.4 we see that  $\mu_a$  can be extended to all of  $\mathbb{R}_+$ .

Hence  $\mu_a$  is a nonnegative global solution to (8) for initial measures in a variation bounded set. On any interval  $J$ , if  $\mu_a$  is a solution to (8), then the set  $\{\mu_a(t) : t \in J\}$  is a bounded set in total variation. Hence we can use Lemma 3.3 along with the Gronwall inequality to show that this solution is unique.

Since  $\vec{0} \in \mathcal{M}_+$ , and  $\mathbb{R}_+ \times (\mathcal{M}_+)_w \times C^{po} = \bigcup_{N \in \mathbb{Z}_+} \mathbb{R}_+ \times (\overline{B_{N,+}(\vec{0})})_w \times C^{po}$ , we let  $\varphi = \bigcup_{N \in \mathbb{Z}_+} \varphi_N$

and Theorem 3.2 is immediate.

## 4 Reduction to Special Cases

Selection and mutation models have been considered on discrete strategy/trait spaces [1, 3, 8] and continuous strategy/trait spaces [10, 11, 30]. In this section we demonstrate the unifying power of the measure theoretic formulation. In particular, we present the correct choices of initial measure  $u$  and the selection-mutation kernel  $\gamma(\hat{q})$  such that the model (8) reduces to each of the cases of interest. Given that our model is nonnegative, we can use Theorem 3.1 and write our model using set function notation.

1. *Reduction to pure selection model:* Let  $\gamma(\hat{q}) = \delta_{\hat{q}}$  and  $u \in \mathcal{M}_+$ . Substituting these parameters in (8) one obtains the pure selection model

$$\begin{cases} \frac{d}{dt}\mu(t; u, \gamma)(E) = \int_E (f_1(\mu(t)(Q), \hat{q}) - f_2(\mu(t)(Q), \hat{q})) d\mu(t)(\hat{q}) \\ \mu(0; u, \gamma) = u. \end{cases} \quad (13)$$

2. *Reduction to density model:* Let  $Q \subset \text{int}(\mathbb{R}_+^n)$  and  $\gamma(\hat{q}), u \in L_1(Q, \nu)$ , i.e, both are absolutely continuous with respect to a measure  $\nu$ . If  $d\gamma(\hat{q}) = P(q, \hat{q})d\nu(q)$  and  $du = c_u(q)d\nu(q)$ , then substituting these expressions into (11) and using Fubini's theorem we see that there exists  $c_0(t, q), f(t, q) \in L^1(Q, \nu)$  such that  $d\varphi(t; u, \gamma) = c_0(t, q)d\nu(q) + f(t, q)d\nu(q) = (c_0(t, q) + f(t, q))d\nu(q)$ . Hence if  $\nu = dq$ , there exists  $x_{u, \gamma}(t, q) \in L^1(\mathbb{R}_+ \times Q, dq)_+$  such that  $d\mu(t) = x_{u, \gamma}(t, q)dq$ . By Fubini's theorem, equation (8) becomes

$$\begin{aligned} \dot{\mu}(E) &= \int_E \dot{x}_{u, \gamma}(t, q)dq \\ &= \int_E \left[ \int_Q f_1(\mu(t)(Q), \hat{q})P(q, \hat{q})x_{u, \gamma}(t, \hat{q})d\hat{q} - f_2(\mu(t)(Q), q)x_{u, \gamma}(t, q) \right] dq, \end{aligned}$$

for all  $E \in \mathcal{B}(Q)$ . Hence

$$\begin{cases} \dot{x}_{u, \gamma}(t, q) = \int_Q f_1(\mu(t)(Q), \hat{q})P(q, \hat{q})x_{u, \gamma}(t, \hat{q})d\hat{q} - f_2(\mu(t)(Q), q)x_{u, \gamma}(t, q) \\ x_{u, \gamma}(0, q) = x_u(q). \end{cases} \quad (14)$$

This is the density replicator-mutator model (4).

3. *Reduction to discrete model:* Assume that  $\gamma, u$  are both discrete, i.e., their support is countable and consists of isolated points. For  $\gamma$  this means that there is a discrete set  $\Lambda$  which contains the support of  $\gamma(\hat{q})$  for all  $\hat{q}$ . Assume there exists a sectionwise continuous function  $P(q, \hat{q})$ , and a family of measures  $\nu(q)$  all having the same discrete support  $\Lambda$  such that  $d\gamma(\hat{q}) = P(q, \hat{q})d\nu(q)$ . Then if we substitute these expressions into equation (11) and use Fubini's theorem we see that there exists  $c_u(t, q), c_\nu(t, q) \in C(\mathbb{R}_+ \times Q)_+$  such that  $d\varphi(t; u, \gamma) = c_u(t, q)du(q) + c_\nu(t, q)d\nu(q)$ . Hence, equation (8) becomes

$$\begin{cases} \dot{\mu} = \int_E \left[ \int_Q f_1(\mu(t)(Q), \hat{q})P(q, \hat{q})c_u(t, \hat{q})du(\hat{q}) \right. \\ \quad \left. + \int_Q f_1(\mu(Q), \hat{q})P(q, \hat{q})c_\nu(t, \hat{q})d\nu(\hat{q}) \right] d\nu(q) \\ \quad - \int_E f_2(\mu(Q), q)c_\nu(t, q)d\nu(q) - \int_E f_2(\mu(Q), q)c_u(t, q)du(q) \\ \mu(0) = u. \end{cases} \quad (15)$$

If  $E = \{q_i\}$  and  $\text{supp}(\nu)$  denotes the support of the  $\nu$ , the above becomes a discrete system given by

$$\begin{cases} \dot{\mu}(\{q_i\}) = \sum_{\hat{q}_j \in \text{supp}\nu} f_1(\mu(t)(Q), \hat{q}_j)P(q_i, \hat{q}_j)[c_u(t, \hat{q}_j) + c_\nu(t, \hat{q}_j) \\ \quad - f_2(\mu(t)(Q), q_i)[c_u(t, q_i) + c_\nu(t, q_i)] \\ \mu(0)(\{q_i\}) = c_u(0, q_i). \end{cases} \quad (16)$$

For example, if  $N = 2, \dots$   $\text{supp}(u) = \{q_i\}_{i=1}^N$ ,  $q = (a(q), b(q))$ ,  $x_i = \mu(\{q_i\})$ ,  $f_1(X, q) = a(q)$ ,  $f_2(X, q) = b(q)X$ ,  $\mu(Q) = X$ , then through a suitable change of variable ( $y_i = a_i x_i$ ) equation (16) reduces to the exact differential equation system studied in [3].

4. Many authors in EGT theory assume that  $f_1$  is a fitness function,  $f_2(X, q) = \int_Q f_1(X, q)d\mu$  is an average fitness, and  $\mu$  is a probability measure. The models are mostly on  $\mathbb{R}_+^n$  and the n-simplex is invariant. From our assumptions we can incorporate a version of this also by using the companion IVP to (8),

$$P(t, u, \gamma)(E) = \frac{\mu(t)(E)}{\mu(t)(Q)}.$$

It has the dynamics

$$\begin{aligned} \frac{d}{dt}P(t; u, \gamma)(E) = & \int_Q \left[ f_1(\mu(t)(Q), \hat{q})\gamma(\hat{q})(E) \right. \\ & \left. - \left( \int_Q f_1(\mu(t)(Q), q)dP(t) \right) P(t)(E) \right] dP(t)(\hat{q}) \\ & - \int_E [f_2(\mu(t)(Q), \hat{q}) - \int_Q f_2(\mu(t)(Q), q)dP(t)] dP(t)(\hat{q}). \end{aligned} \quad (17)$$

5. *Reduction to Density Dependent Replicator Equation:* Define

$$f(\mu(Q), q) = f_1(\mu(Q), q) - f_2(\mu(Q), q).$$

If  $\pi(dq, \mu) = F(\mu)(dq) = f(\mu(Q), q)\mu(dq)$ , then (17) becomes

$$\dot{P}(t)(E) = \int_E [f(X, q) - \bar{f}(X, q)]dP(q),$$

where  $\bar{f} = \int_Q f d\mu$ . This is exactly the density dependent Replicator equation.

6. *Reduction to Density Dependent Quasi-species Equation:* Likewise if we interpret  $f_1(\mu(Q), q)$  as a net fitness and  $f_2(\mu) = \int_Q f_1(\mu(Q), q)d\mu$  as the average fitness, then (8) becomes the Density Dependent Replicator-Mutator equation.

## 5 Concluding Remarks

We have formulated a density dependent EGT (selection-mutation) model on the space of measures and provided a framework which is rich enough to allow pure selection, selection-mutation, and discrete and continuous strategy spaces, all under one setting. We also established the well-posedness of this EGT model.

There are several future paths to take from this point. We will mention one application and one mathematical future pathway. Modeling tumor growth, cancer therapy and viral evolution are immediate applications. For example, tumor heterogeneity is one main cause of tumor robustness. Tumors are robust in the sense that tumors are systems that tend to maintain stable functioning despite various perturbations. While tumor heterogeneity describes the existence of distinct subpopulations of tumor cells with specific characteristics within a single neoplasm. The mutation between the subpopulations is one major factor that makes the tumor robust. To date there is no unifying framework in mathematical modeling of carcinogenesis that would account for parametric heterogeneity [18]. To introduce distributed parameters (heterogeneity) and mutation is essential as we know that cancer recurrence, tumor dormancy and other dynamics can appear in heterogeneous settings and not in homogeneous settings. Increasing technological sophistication has led to a resurgence of using oncolytic viruses in cancer therapy. So in formulating a cancer therapy it is useful to know that in principle *a heterogeneous oncolytic virus must be used to eradicate a tumor cell*.

One mathematical future path is to perform asymptotic analysis on the model. There are two essential things that need to be addressed if we wish to be able to perform asymptotic analysis of our model. First, we need a state space with the property that if the measure valued dynamical system has an initial condition as a finite signed Borel measure then the asymptotic limits will also be in this space. The second problem is that often there will be more than one strategy of a given fitness. In (1), a Dirac mass emerged as it is assumed that only a unique fittest class exists. In reality, this may not be the case and more than one fittest class can exist. In particular, it is possible that a continuum of fittest strategies exist (see Figure 1 for an example). So our mathematical structure must include the ability to demonstrate the convergence of the model solution to a measure supported on a continuum of strategies.

These two difficulties coupled with our desire to study the problem of parameter estimation in these models imply that some form a “weak” or “generalized” asymptotic limit must

be formulated. These weak limits need to live in a certain “completion” of the space of finite signed measures. We will explore these topics in a forthcoming study.

**Insert Figure 1 Here**

**Acknowledgements:** The authors would like to thank Horst Thieme for thorough reading of an earlier version of this manuscript and for the many useful comments. The authors would also like to thank their colleague Ping Ng for helpful discussions. This work was partially supported by the National Science Foundation under grant # DMS-0718465.

## References

- [1] A.S. Ackleh and L.J.S. Allen, *Competitive exclusion and coexistence for pathogens in an epidemic model with variable population size*, J. Math. Biol., **47** (2003), 153-168.
- [2] A.S. Ackleh, B.G. Fitzpatrick and H.R. Thieme, *Rate Distributions and Survival of the Fittest: A Formulation on the Space of Measures*, Discrete Contin. Dyn. Syst. Ser. B, **5** (2005), 917-928.
- [3] A.S. Ackleh and S. Hu, *Comparison between stochastic and deterministic selection-mutation models*, Math. Biosci. Eng., **4** (2007), 133-57.
- [4] A.S. Ackleh, D.F. Marshall, H.E. Heatherly, and B.G. Fitzpatrick, *Survival of the fittest in a generalized logistic model*, Math. Models Methods Appl. Sci., **9** (1999), 1379-1391.
- [5] C.D. Aliprantis and K.C. Border, *Infinite Dimensional Analysis*, Springer-Verlag, 1994
- [6] I. M. Bomze, *Cross Entropy minimization in uninvadable states of complex populations*, J. Math. Biol. **30** (1991), 73-87.
- [7] N. Bourbaki, *Integration I*, Berlin-New York, Springer-Verlag, 2004.
- [8] H. J. Bremermann and H.R. Thieme, *A competitive exclusion principle for pathogen virulence*, J. Math. Biol., **27**, (1989), 179-190.
- [9] J.S. Brown and B.J. McGill, *Evolutionary game theory and adaptive dynamics of continuous traits*, Ann. Rev. Ecol. Evol. Syst., **38** (2007), 403-435.
- [10] A. Calsina and S. Cuadrado, *Small mutation rate and evolutionarily stable strategies in infinite dimensional adaptive dynamics*, J. Math. Biol., **48** (2004), 135-159.

- [11] A. Calsina and S. Cuadrado, *Asymptotic stability of equilibria of selection mutation equations*, J. Math. Biol., **54** (2007), 489-511.
- [12] V. Chellaboina and W.M. Haddad, *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach*, Princeton University Press, 2008.
- [13] R. Cressman and J.Hofbauer, *Comparison between stochastic and deterministic selection-mutation models*, Theor. Popul. Biol., **67** (2005), 47-59.
- [14] M. Eigen, J. McCaskill and P. Schuster, *The molecular quasi-species*, Adv. Chem. Phys., **75** (1989), 149-263.
- [15] S. Genieys, V. Volpert and P. Auger, *Pattern and waves for a model in population dynamics with nonlocal consumption of resources*, Math. Model. Nat. Phenom., **1**, (2006), 6582.
- [16] J. Hofbauer, *The Selection mutation equation*, J. Math. Biol., **23** (1985), 41-53.
- [17] J. Hofbauer and K. Sigmund, *The Theory and Evolution of Dynamical Systems*, Cambridge University Press, (1998).
- [18] G. P. Karev, A.S. Novozhilov, E.V. Koonin, *Mathematical modeling of tumor therapy with oncolytic viruses: effects of parametric heterogeneity on cell dynamics*, Biology Direct, **1** (2006), 1-19.
- [19] Y.G. Kondratiev, T.Kuna, N. Ohlerich, *Selection-mutation balance models with epistatic selection*, Condens. Matter Phys., **11** (2008), 283-291.
- [20] S. Lang, *Undergraduate Analysis*, Secaucus, New Jersey, Springer Verlag, 1983.
- [21] V. Lakshmikantham and S. Leela, *An Introduction to Nonlinear Differential Equations in Abstract Spaces*, Pergamon Press, Oxford, 1981.
- [22] P. Magal, *Mutation and recombination in a model of phenotype evolution*, J. Evol. Equ., **2** (2002), 21-39.
- [23] P. Magal and G.F. Webb, *Mutation, selection and recombination in a model of phenotype evolution*, Discrete Contin. Dyn. Syst., **6** (2000), 221-236.
- [24] M. A. Nowak, *Evolutionary Dynamics*, Belknap Press, 2006.

- [25] J. Maynard Smith and G.R. Price *The logic of animal conflict*, Nature **246** (1973), 15-18.
- [26] B. Perthame, *Transport Equation in Biology*, Frontiers in Mathematics series, Birkhauser, 2005.
- [27] G. Raoul, *Local stability of evolutionary attractors for continuous structured populations*, accepted in Monatshefte fur Mathematik.
- [28] G. Raoul, *Long time evolution of populations under selection and vanishing mutations*, Acta Applicandae Mathematica, **114**, (2011), 1-14.
- [29] H. Royden, *Real Analysis: Third Edition*, Prentice Hall, 1988.
- [30] J. Saldana, S.F. Elena, and R.V. Sole, *Coinfection and superinfection in RNA virus populations: a selection-mutation model*, Math. Biosci., **183** (2003), 135-160.

## 6 Appendix

For the convenience of the reader, we next state a few known results that are used in our analysis.

**Theorem 6.1.** (*Differentiation Under Integral*) If  $a \leq \alpha \leq b$ , let

$$\phi(\alpha) = \int_{u_1(\alpha)}^{u_2(\alpha)} f(x, \alpha) dx.$$

Then  $\phi_\alpha = \int_{u_1(\alpha)}^{u_2(\alpha)} f_\alpha(x, \alpha) dx + f(u_2, \alpha)u_{2,\alpha} - f(u_1, \alpha)u_{1,\alpha}$  provided  $f, f_\alpha$  are continuous in some region containing  $\{x | u_1 \leq x \leq u_2\} \times (a, b)$  and  $u_1, u_2 \in C^1(a, b)$ .

**Definition 6.2.** [5, pg. 151] A dual pair or a dual system is a pair  $\langle X, X' \rangle$  of vector spaces over a field  $F$  together with a function  $(x, x') \mapsto \langle x, x' \rangle \in F$  satisfying the following:

1. The map  $x \mapsto \langle x, x' \rangle$  is linear for each  $x'$ .
2. If  $\langle x, x' \rangle = 0$  for each  $x'$ , then  $x = 0$ .
3. The map  $x' \mapsto \langle x, x' \rangle$  is linear for each  $x$ .
4. If  $\langle x, x' \rangle = 0$  for each  $x$ , then  $x' = 0$ .

Each space of a dual pair  $\langle X, X' \rangle$  can be interpreted as a set of linear functionals on the other. For instance, each  $x \in X$  defines the linear functional  $x' \mapsto \langle x, x' \rangle$ . If  $A \subseteq X$ , then it is called  $X'$  – *bounded* if  $\sup_{x \in A} |\langle x, x' \rangle|$  is bounded for every  $x' \in X'$ . For each  $X'$  – *bounded* subset  $A \subseteq X$  we define the semi-norm on  $X'$

$$p_A(x') = \sup_{x \in A} |\langle x, x' \rangle|.$$

If  $\beta$  is a system of  $X'$  – *bounded* subsets, the family  $\{p_A | A \in \beta\}$  generates a Hausdorff locally convex topology called the  $\beta$  topology. A net  $(x'_\alpha)$  converges to  $x'$  iff  $p_A(x'_\alpha - x') \rightarrow 0$  for all  $A \in \beta$ . If  $\beta$  consists of singletons, then it is called the weak\* topology on  $X'$  and is often also denoted as  $\sigma(X', X)$ .

**Theorem 6.3.** [5, pg. 153] (*Duality pairs are weakly dual*) Let  $X, Y$  be topological vector spaces over a field  $F$  forming a dual pair. The topological dual of  $(X, \sigma(X, Y))$  is  $Y$ . Similarly  $(Y, \sigma(Y, X))' = X$ . Here  $\sigma(X, Y)$  ( $\sigma(Y, X)$ ), denotes the weak topology on  $X$  generated by the family of linear functionals  $\{\langle \cdot, y \rangle\}_{y \in Y}$ . Also  $X'$  is the notation used for the continuous dual of  $X$ .

The next theorem is concerned with

$$x' = f(t, x), \quad x(t_0) = x_0, \tag{18}$$

where  $f \in C[\mathbb{R}_+ \times E, E]$ ,  $E$  being a Banach space.

**Theorem 6.4.** [21, pg. 145] Assume that

$$\|f(t, x)\| \leq g(t, \|x\|), \quad (t, x) \in \mathbb{R}_+ \times E,$$

where  $g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$ ,  $g(t, u)$  is nondecreasing in  $u$  for each  $t \in \mathbb{R}_+$ , and the maximal solution  $r(t, t_0, u_0)$  of the scalar differential equation

$$u' = g(t, u), \quad u(t_0) = u_0 \geq 0,$$

exists on  $[t_0, \infty)$ . Suppose that  $f$  is smooth enough to assure local existence of solutions to (18) for any  $(t_0, x_0) \in \mathbb{R}_+ \times E$ . Then the largest interval of existence of any solution  $x(t, t_0, x_0)$  of (18) such that  $\|x_0\| \leq u_0$  is  $[t_0, \infty)$ . If in addition  $r(t, t_0, u_0)$  is bounded, then  $\lim_{t \rightarrow \infty} x(t, t_0, x_0) = y \in E$ .



**Definition 6.5.** [7, III.33] Let  $X$  be locally compact,  $E$  a Hausdorff locally convex space, and  $\mu$  a measure on the Borel sets of  $X$ . For every  $f \in C_c(X; E)$  we call the integral of  $f$  with respect to  $\mu$ ,  $\int f d\mu$ , the element of  $E'^{\sharp}$  where  $E'$  is the continuous dual and  $E'^{\sharp}$  is the algebraic dual defined by

$$\left\langle \int f d\mu, z' \right\rangle = \int_X \langle f(x), z' \rangle d\mu(x), \quad \text{for all } z' \in E'.$$

**Theorem 6.6.** [7, III.37] Let  $X$  be as in Definition 6.5, and let  $\mathcal{B}_X$  denote the Borel sets on  $X$ . Suppose  $f$  is a continuous mapping with compact support of  $(X, \mathcal{B}_X)$  into a Hausdorff locally convex space  $E$  and  $q$  is a continuous semi-norm on  $E$ . Then for every measure  $\mu$  on  $(X, \mathcal{B}_X)$  such that  $\int f d\mu \in E$ ,

$$q\left(\int f d\mu\right) \leq \int (q \circ f) d|\mu|.$$

**Theorem 6.7.** [7, III.37] Let  $X$  be as in Definition 6.5, let  $E$  be a Hausdorff locally convex space, and  $f \in C_c(X; E)$ . If  $f(X)$  is contained in a complete convex subset  $A$  of  $E$ , then  $\int f d\mu \in E$ .

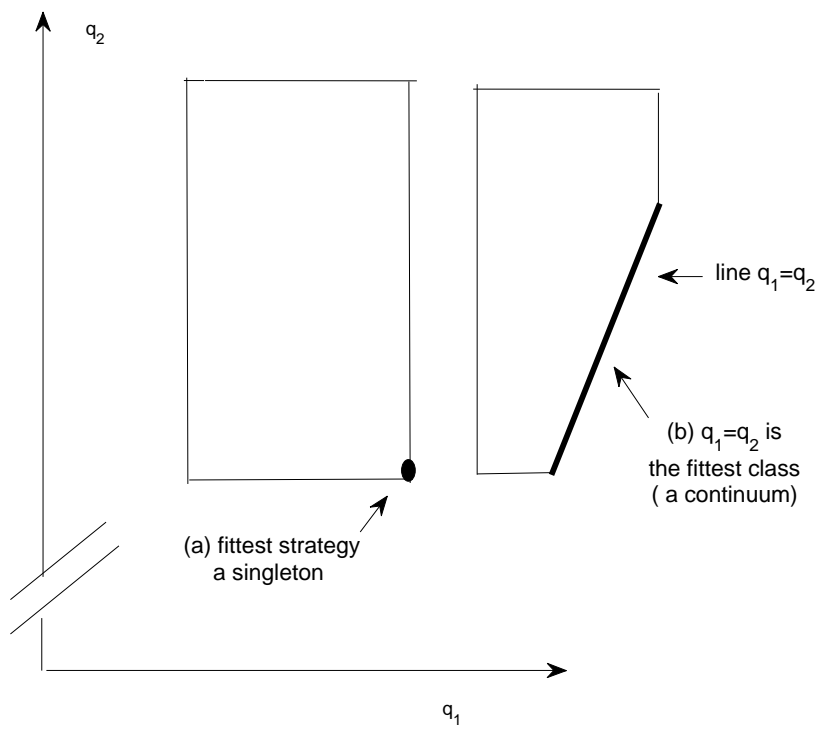


Figure 1: Two examples of strategy spaces.

